Competitive Equilibrium and the Welfare Theorems

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- Rather than having a social planner, set up a market structure with firms (who maximize profits) and households (who maximize utility).
    - time-0 market structure
    - sequential market structure
- The Two Welfare Theorems
    - 1st Welfare Theorem: circumstances under which a competitive equilibrium is Pareto optimal (i.e. it corresponds to the solution to a social planning problem).
    - 2nd Welfare Theorem: circumstances under which a Pareto optimum (the solution to a social planning problem) can be supported as a competitive equilibrium.
- Stochastic Models

Setting up the Market Structure
Households

- Many market structures are possible: we will look at two examples.
- Households own all the factors of production and shares in the firms.
    - Endowments of factors and assets are distributed equally across households–allows us to abstract from trade in the assets
- Households sell factor services (labor and capital) to firms
- Households use their income to either consume or accumulate more capital.
- Households wish to maximize lifetime utility

- Firms own nothing, hire factors of production to produce output which they sell to households.
- Profits are distributed to owners.
- Since the firm's problem is not dynamic the firm's goal is to maximize profits.

Setting up the Market Structure

Markets and Trade: A Time 0 Structure

- We will look first at the time 0 market structure—come back to the sequential market structure at the end.
- Trading and pricing of contracts all takes place at time 0, determining the future sequences of prices and quantities.
- After time 0 there is no more trade, simply the delivery of the services and goods promised under the contracts drawn up at time 0.
- p_t: time-0 price of a unit of output delivered at time t in an arbitrary unit of account
- w_t: price of a unit of labor delivered in period t expressed in units of goods delivered in period t (real wage)
- r_{kt}: rental rate of capital in units of goods delivered in period t.

The Firm's Problem

- The firm chooses $\{y_t, k_t^d, L_t^d\}_{t=0}^{\infty}$, to maximize

$$\Pi = \sum_{t=0}^{\infty} p_t (y_t - r_{kt} k_t^d - w_t L_t^d)$$

subject to $y_t \le F(k_t^d, L_t^d)$, $t \ge 0$, and taking the sequences $\{p_t, w_t, r_{kt}\}_{t=0}^{\infty}$ as given.

- Equivalent to a sequence of static problems where the firm maximizes $y_t - r_{kt} k_t^d - w_t L_t^d$.

The Household's Problem

- Taking the price sequences $\{p_t, w_t, r_{kt}\}_{t=0}^{\infty}$ as given and the firm's profits, Π, as given the household maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t(c_t + i_t) \le \sum_{t=0}^{\infty} p_t(r_{kt}k_t^s + w_t L_t^s) + \Pi$$

$$k_{t+1} = (1-\delta)k_t + i_t,\ t \ge 0$$

$$0 \le L_t^s \le 1, \quad 0 \le k_t^s \le k_t,\ t \ge 0$$

$$c_t \ge 0,\ k_{t+1} \ge 0,\ t \ge 0.$$

Formal Description of Competitive Equilibrium

A *competitive equilibrium* is a set of prices $\{p_t, r_{kt}, w_t\}_{t=0}^{T}$, and allocations $\{k_t^d, L_t^d, y_t\}_{t=0}^{\infty}$ and $\{c_t, I_t, k_{t+1}, k_t^s, L_t^s\}_{t=0}^{\infty}$ for firms and households, respectively, such that

- $\{k_t^d, L_t^d, y_t\}_{t=0}^{\infty}$ solves the firm's problem given $\{p_t, r_{kt}, w_t\}_{t=0}^{\infty}$,
- $\{c_t, I_t, k_{t+1}, k_t^s, L_t^s\}_{t=0}^{\infty}$ solves the household's problem given $\{p_t, r_{kt}, w_t\}_{t=0}^{\infty}$, and Π,
- all markets clear: $k_t^d = k_t^s$, $L_t^d = L_t^s$, $c_t + i_t = y_t$, $t \geq 0$.

- Conjecture that p_t, w_t, r_{kt} are strictly positive for all t.
- The firm, essentially, has a sequence of static problems.
  - For each t, given $p_t > 0$ it picks k_t^d and L_t^d to maximize $F(k_t^d, L_t^d) - r_{kt} k_t^d - w_t L_t^d$. Hence

$$
\begin{aligned}
r_{kt} &= F_k(k_t^d, L_t^d) && (1)\\
w_t &= F_n(k_t^d, L_t^d). && (2)
\end{aligned}
$$

- Since F is CRTS it follows that $F(k_t^d, L_t^d) - r_{kt} k_t^d - w_t L_t^d = 0$, $\forall t$, and therefore that $\Pi = 0$.

- Optimal for the household to set $L_t^s = 1$ and $k_t^s = k_t$.
- Budget constraint will always hold with equality, given the properties of u.
- Rewrite the household's problem as

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ subject to}$$

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\delta)k_t\right] = \sum_{t=0}^{\infty} p_t (r_{kt} k_t + w_t) + \Pi \tag{3}$$

$$c_t \geq 0,\ k_{t+1} \geq 0,\ t \geq 0.$$

- Nonnegativity constraint on c_t never holds with equality, so:

$$\beta^t u'(c_t) - \theta p_t = 0, \; t \geq 0 \tag{4}$$
$$\theta[(r_{kt+1} + 1 - \delta)p_{t+1} - p_t] \leq 0, \; t \geq 0, \tag{5}$$

where θ is the Lagrange multiplier on the budget constraint.

  - The inequality is an equality for any t such that $k_{t+1} > 0$ (assume that $k_{t+1} > 0$ for all t).

- Imposing the equilibrium conditions $k_t^d = k_t^s = k_t$, $L_t^d = L_t^s = 1$ and $c_t + i_t = y_t$, and using (1) we can rewrite (4) and (5) as

$$\begin{aligned} \beta^t u'(c_t) &= \theta p_t, \; t \geq 0 \\ [f'(k_{t+1}) + 1 - \delta]p_{t+1} &= p_t, \; t \geq 0 \end{aligned}$$

and we also have

$$c_t + k_{t+1} - (1 - \delta)k_t = f(k_t), \; t \geq 0.$$

The Two Welfare Theorems

- Notice that if we substitute p_t out of our equilibrium conditions we have

$$\begin{aligned} \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta] &= u'(c_t),\ t \geq 0 \\ c_t + k_{t+1} - (1-\delta)k_t &= f(k_t),\ t \geq 0. \end{aligned}$$

- These are the same as the optimality conditions from the social planner's problem.
- Although this is not a formal proof of the two welfare theorems, we have constructed a competitive equilibrium which is characterized by the same conditions as the social planner's problem. Thus we have shown that
  - the competitive equilibrium is pareto optimal
  - that we can support the social planner's solution with this competitive equilbrum

- We can consider an alternative market structure in which agents trade contracts in each period.
- Write prices and single-period profits as functions of the state variables, so that they can be represented in a dateless formulation of the household's problem
  - $r_{kt} = r_k(k_t)$, $w_t = w(k_t)$, $\pi_t = \pi(k_t)$
- Continue to assume household supplies labor inelastically
- Continue to abstract from trade in shares of the firms.
- Could add trade in single period securities that pay a unit of consumption in the next period to show comparability to time 0 market structure
- The firm's problem remains the same because it is static.

- Let K and C be the household's own capital and consumption, k the aggregate capital stock, which is the state variable.
- Household solves

$$V(K,k) = \max_{C,K'} \left\{ u(C) + \beta V\left[K', h(k)\right] \right\} \tag{6}$$

subject to

$$C + K' - (1-\delta)K \leq K r_k(k) + w(k) + \pi(k)$$

A *recursive competitive equilibrium* is a value function, V, a policy function for the household, H, a law of motion for the aggregate capital stock, h, and functions r, w and π, such that

- V satisfies (6),
- H is the optimal policy function for (6),
- $H(k, k) = h(k)$ for all k,
- $r_k(k)$ and $w(k)$ satisfy the firm's first order conditions; i.e.

$$r_k(k) = F_k(k, 1) \text{ and } w(k) = F_n(k, 1)$$

- $\pi(k) = F(k, 1) - r_k(k)k - w(k)$.

- The first order conditions for the firm's problem are the same as before

$$r_k(k) = F_k(k^d, L^d) \text{ and } w(k) = F_n(k^d, L^d)$$

- In equilibrium we must have $k^d = k$ and $L^d = 1$ so that

$$r_k(k) = F_k(k, 1) \text{ and } w(k) = F_n(k, 1)$$

- The firm's profits single period profits are

$$\pi = F(k^d, L^d) - r_k(k)k^d - w(k)L^d$$

- In equilibrium profits are zero from CRTS and the fact that $k^d = k$ and $L^d = 1$.
- Hence $\pi(k) = 0$ for all k.

Solving for the Recursive Equilibrium

The Household's Problem

- After substituting out C, the first-order and envelope conditions for the household are

$$u'(C) = \beta V_1\left[K', h(k)\right]$$

$$V_1(K,k) = u'(C)\left[r_k(k) + 1 - \delta\right]$$

- Combining these we have the usual Euler equation and the budget constraint

$$u'(C) = \beta u'\left(C'\right)\left[r_k(k') + 1 - \delta\right]$$

$$C + K' - (1-\delta)K \leq K r_k(k) + w(k) + \pi(k)$$

- Imposing $C = c$ and $K = k$, and given the results from the firm's problem which determined $r_k(k)$, $w(k)$ and $\pi(k)$ we have

$$u'(c) = \beta u'\left(c'\right)\left[f'(k') + 1 - \delta\right]$$

$$c + k' - (1-\delta)k \leq f(k)$$

- This is equivalent to what we got from the time 0 structure.

Solving for the Recursive Equilibrium

What Would the Bonds have Added?

- If we had allowed households to trade single period bonds we would have had to modify the budget constraint to be:

$$C + K' - (1-\delta)K + q(k,b)B' \leq Kr_k(k,b) + w(k,b) + \pi(k,b) + B.$$

- Since the aggregate quantity of bonds must be $b = 0$ in equilibrium, the first-order and envelope conditions for B' would have been

$$q(k,0)u'(C) = \beta V_2\left[K', B', h(k,b), 0\right]$$

$$V_2\left[K, B, k, 0\right] = u'(C).$$

- Hence the price of a one period bond is

$$q(k,0) = \beta u'(C')/u'(C)$$

- At date t, q_t is the same as p_{t+1}/p_t from the time 0 problem.

Alternative Market Arrangements

- There are many possible market arrangements that we have not explored that would lead to equivalent outcomes
- An important case is when the households do not own the capital stock, and instead it is owned by firms who also make the investment decisions.
- With this setup the firms and the households both have dynamic problems, and it is critical to allow the households to trade the one period bonds
- Firms have to discount their profit flow, and do so using the prices of the bonds.
    - This ensures that the firms choose the same investment the household would have

- We described a model in which output per capita is $z_t f(k_t)$.
- To set up a market structure we need to be formal and write

$$z_t = z_t(s^t)$$

where s^t is the *history* of a stochastic *event* s_t up to date t. I.e.

$$s^t = (s_t, s_{t-1}, \ldots, s_0).$$

- Unconditional probability of observing a particular history is

$$\pi_t(s^t)$$

- Also have conditional probabilities

$$\pi_\tau(s^\tau | s^t)$$

- Assume that s_0 is known.

The Social Planner's Problem in the Stochastic Model

Basic Setup

- Recall that the social planner maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)]$$

s.t. $c_t = z_t f(k_t) + (1-\delta)k_t - k_{t+1}$, for $t \geq 0$, and k_0 given.

- The planner has to choose contingency plans—choices of the future k_ts that are contingent on realizations of the state.
  - The planner chooses $c_t(s^t), k_{t+1}(s^t)$ for each t and each possible s^t.
- Assuming a discrete distribution for the shocks, this can be rewritten as

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u[c_t(s^t)]$$

s.t. $c_t(s^t) = z_t(s^t) f[k_t(s^{t-1})] + (1-\delta)k_t(s^{t-1}) - k_{t+1}(s^t)$ for each t and s^t.

The Social Planner's Problem in the Stochastic Model

The Lagrangian

- Abstracting from issues arising from infinite numbers of choice variables form the Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \left(u\left[c_t(s^t)\right] + \mu_t(s^t) \left\{ z_t(s^t) f\left[k_t(s^{t-1})\right] + \cdots \right.\right.$$

$$\left.\left. (1-\delta)k_t(s^{t-1}) - k_{t+1}(s^t) - c_t(s^t) \right\} \right)$$

- The first order conditions are

$$\begin{aligned} u'[c_t(s^t)] &= \mu_t(s^t) \\ \beta^t \pi_t(s^t)\mu_t(s^t) &= \sum_{s^{t+1}|s^t} \beta^{t+1} \pi_{t+1}(s^{t+1}) \mu_{t+1}(s^{t+1}) \times \\ &\qquad \left\{ z_{t+1}(s^{t+1}) f'[k_{t+1}(s^t)] + (1-\delta) \right\} \end{aligned}$$

- Rewritten these become the familiar Euler equation

$$u'[c_t(s^t)] = \sum_{s^{t+1}|s^t} \beta\pi_{t+1}(s^{t+1}|s^t)u'[c_{t+1}(s^{t+1})] \times \left\{z_{t+1}(s^{t+1})f'[k_{t+1}(s^t)] + (1-\delta)\right\}$$

or

$$u'(c_t) = E_t\beta u'(c_{t+1})[z_{t+1}f'(k_{t+1}) + (1-\delta)].$$

- This is the same as the Euler equation we got in the notes on dynamic programming.
- Now we want to show equivalence of the social planning problem to a competitive equilibrium.

- The firm maximizes

$$\Pi = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) \left\{ z_t(s^t) F\left[k_t^d(s^t), L_t^d(s^t) \right] - r_{kt}(s^t) k_t^d(s^t) - w_t(s^t) L_t^d(s^t) \right\}$$

- Firm's problem is fundamentally static:

$$\begin{aligned} r_{kt}(s^t) &= z_t(s^t) F_k\left[k_t^d(s^t), L_t^d(s^t) \right] \\ w_t(s^t) &= z_t(s^t) F_n\left[k_t^d(s^t), L_t^d(s^t) \right] \end{aligned}$$

- CRTS technology implies zero profits.

The Decentralized Model

The Household's Problem

- The household maximizes

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)u[c_t(s^t)]$$

s.t.

$$\sum_{t=0}^{\infty}\sum_{s^t}p_t(s^t)\left[c_t(s^t)+k_{t+1}(s^t)-(1-\delta)k_t(s^{t-1})\right]\leq$$

$$\sum_{t=0}^{\infty}\sum_{s^t}p_t(s^t)\left[r_{kt}(s^t)k_t^s(s^t)+w_t(s^t)L_t^s(s^t)\right]+\Pi$$

- The household will set $L_t^s(s^t) = 1$ for all t, s^t and $k_t^s(s^t) = k_t(s^{t-1})$ for all t, s^t.
- The household's first order conditions for $c_t(s^t)$ and $k_{t+1}(s^t)$ are

$$\beta^t \pi_t(s^t) u'[c_t(s^t)] = p_t(s^t)$$

$$p_t(s^t) = \sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1}) \left[r_{kt+1}(s^{t+1}) + (1-\delta) \right]$$

- Substituting out $p_t(s^t)$ and using $r_{kt}(s^t) = F_k\left[k_t(s^{t-1}), 1\right] = f'[k_t(s^{t-1})]$ we have

$$
\begin{aligned}
u'[c_t(s^t)] &= \sum_{s^{t+1}|s^t} \beta\pi_{t+1}(s^{t+1}|s^t)u'[c_{t+1}(s^{t+1})] \times \\
&\quad \left\{z_{t+1}(s^{t+1})f'[k_{t+1}(s^t)] + (1-\delta)\right\}
\end{aligned}
$$

  - This is just the Euler equation again!

- We also impose market clearing in the goods market,

$$
c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1}) = z_t(s^t)f\left[k_t(s^{t-1})\right]
$$

which guarantees that we replicate the social planner problem.

The Sequential Markets Decentralized Model

- As you will see if you try to read Ljunqvist-Sargent, formulating the sequential markets representation of the decentralized economy is hideous unless you assume that s_t is a Markov process
- Since we did this when thinking about the social planning problem in the previous set of slides, we will immediately go the Markov case here.
- We will use the big K-little k trick we used earlier in this chapter to represent household/firm choices versus aggregate variables

- The representative firm's problem remains fundamentally static. It maximizes

$$\pi(k,s) = \max_{K^d, L^d} z(s) F\left(K^d, L^d\right) - r_k(k,s) K^d - w(k,s) L^d$$

- First order conditions:

$$\begin{aligned} r_k(k,s) &= z(s) F_k\left(K^d, L^d\right) \\ w(k,s) &= z(s) F_n\left(K^d, L^d\right) \end{aligned}$$

- CRTS technology implies $\pi(k,s) = 0$ for all k, s.
- The firm's problem determines K^d and L^d as functions of the current aggregate states, k and s.

- The household's problem, which is recursive, can be represented by the following Bellman equation

$$V(K, k, s) = \max_{C, K'} u(C) + \beta \sum_{s'} V\left[K', h(k, s), s'\right] \pi(s'|s)$$

subject to

$$C + K' - (1 - \delta)K \leq K r_k(k, s) + w(k, s) + \pi(k, s)$$

- If we substitute in the constraint and differentiate with respect to K' we get

$$u'(C) = \beta \sum_{s'} V_1 \left[K', h(k, s), s'\right] \pi(s'|s)$$

- The envelope condition is

$$V_1(K, k, s) = u'(C)[r_k(k, s) + (1 - \delta)]$$

- Combining these we have

$$u'(C) = \beta \sum_{s'} u'(C') \left\{r\left[h(k, s), s'\right] + (1 - \delta)\right\} \pi(s'|s)$$

which is the same old Euler equation.

- Imposing market clearing we have

$$\begin{aligned} r_k(k,s) &= z(s)F_k(k,1) \\ w(k,s) &= z(s)F_n(k,1) \end{aligned}$$

and

$$c(k,s)+h(k,s)-(1-\delta)k = z(s)f(k)$$

with the Euler equation becoming

$$u'[c(k,s)] = \beta \sum_{s'} u'\{c[h(k,s),s]\}\{z(s')f'[h(k,s)]+(1-\delta)\}\pi(s'|s)$$

- Once again, the decentralized economy replicates the social planning solution.